Sydney Technical High School

Mathematics

H.S.C. ASSESSMENT TASK 3

JUNE 2012

General Instructions

- Working Time – 70 minutes.
- Approved calculators may be used.
- A table of Standard Integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a new side of the answer booklet.
- Marks shown are a guide and may need to be adjusted.
- Full marks may not be awarded for careless work or illegible writing.
- **For Questions1-5, write the letter for the correct answer on the first page of your answer booklet. Be very clear.**

NAME ____________________

TEACHER ____________________

Question 1

The sine curve with period 4π units and amplitude 2 units has equation:

A. $y = 2\sin\frac{x}{2}$ B. $y = 2\sin\frac{x}{4}$ C. $y = 4\sin 2x$ D. $y = 4\sin\frac{x}{2}$ E. $y = 2\sin 4x$

Question 2

The derivative of sin^2x is :

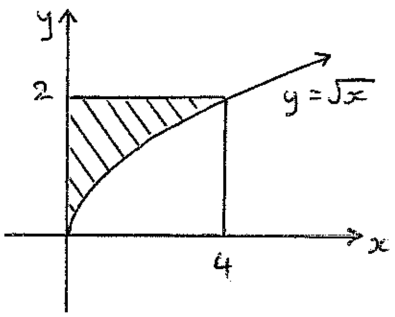
A. cos^2x B. $2\sin x$ C. $2\cos x$ D. $2\cos x\sin x$ E. none of these.

Question 3

The primitive of cos^2x is :

A. sin^2x B. $\frac{cos^3x}{3}$ C. $\frac{sin^3x}{3}$ D. $\frac{cos^3x}{3\sin x}$ E. none of these.

Question 4



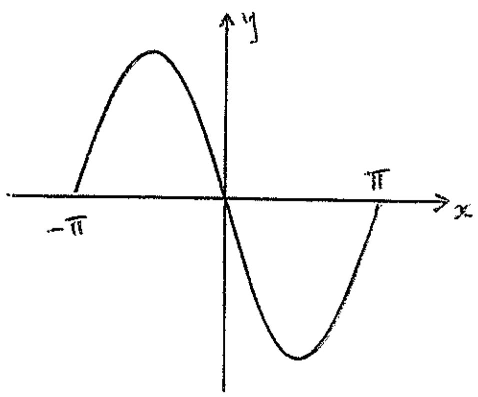
The shaded area can be found using :

A. $\int_0^2 \sqrt{x}\,dx$ B. $\int_0^2 y\,dy$ C. $\int_0^2 y^2\,dy$

D. $\int_0^4 y^2\,dy$ E. $\int_0^4 \sqrt{x}\,dx$

Question 5

Which of the following is NOT a possible function for the curve shown :



A. $y = -\sin x$ B. $y = \sin(x+\pi)$

C. $y = \sin(x-\pi)$ D. $y = \cos(x-\frac{\pi}{2})$

E. $y = \cos(x+\frac{\pi}{2})$

Question 6 (12 marks) Start on a new page. Marks

a) Convert $\frac{\pi}{10}$ radians to degrees. 1

b) Give the exact value of $\operatorname{cosec} \frac{\pi}{4}$ 1

c) Solve $tan^2 x - \tan x = 0$ for $0 \leq x \leq 2\pi$ 3

d) Find the gradient of the tangent to the curve $y = 3\sin 2x$ at the point where $x = \frac{\pi}{12}$ 2

e) Evaluate $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} sec^2 2x\, dx$. Leave your answer in exact form. 2

f) Find the total area between the curve $y = \sin x$ and the x axis for $\frac{-3\pi}{2} \leq x \leq \frac{3\pi}{2}$ 2

g) Evaluate $\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \tan x\, dx$ 1

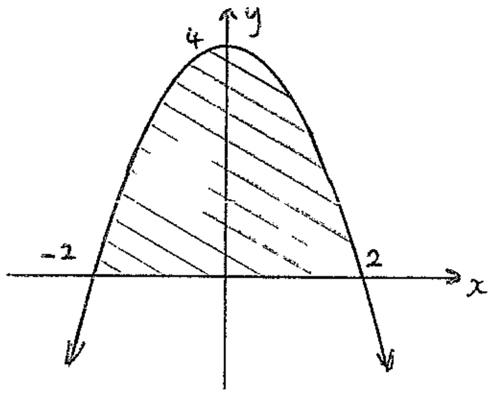
Question 7 (14 marks) Start on a new page.

a) i) Find $\frac{d}{dx}(tan^2 x)$ 1

ii) Hence find $\int \tan x\, sec^2 x\; dx$ 1

b) Find $\frac{d}{dx}(cos^3 5x)$ 2

c) The graph of $y = 4 - x^2$ is shown :



i) Use the Trapezoidal Rule and 5 function values to approximate the shaded area above. 2

ii) Find the exact value of the shaded area. 3

iii) The shaded area is rotated about the y-axis. Find the generated volume in exact form. 3

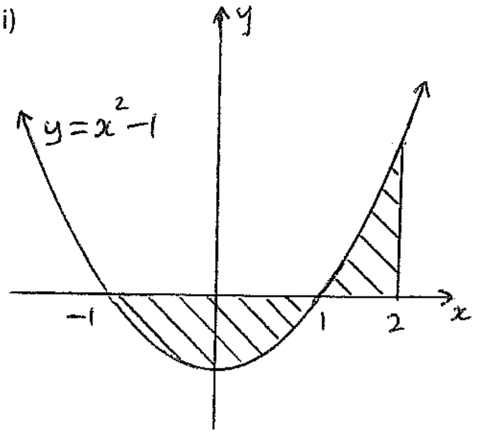
d) Find an angle, x radians, such that the gradient on the curve $y = \tan x$ has value 2. 2

Question 8 (12 marks) Start on a new page.

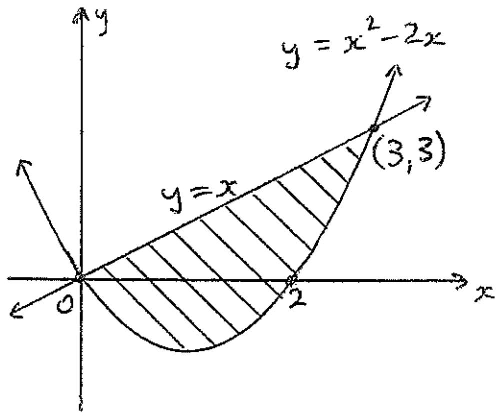
a) Write an integral expression that represents the total shaded area of each situation below :

DO NOT EVALUATE THE INTEGRALS.

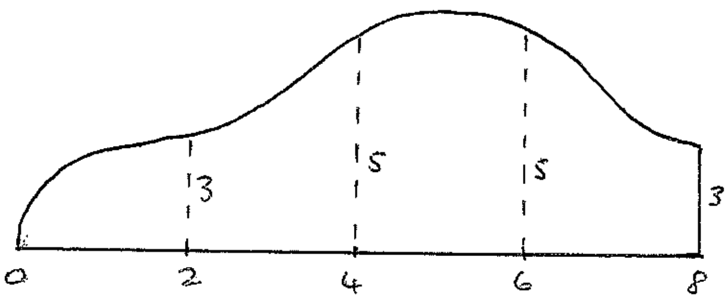
i) 2



ii) 2



b) The cross-sectional area of a rock wall is shown. Horizontal lengths and their corresponding vertical heights are indicated, in metres.



Find the approximate area above using Simpson's Rule and 5 function values. 2

c) i) Sketch the curve $y = 2\cos 4x$ for $0 \le x \le \frac{\pi}{4}$. Use a ruler and clearly label x, y intercepts. 2

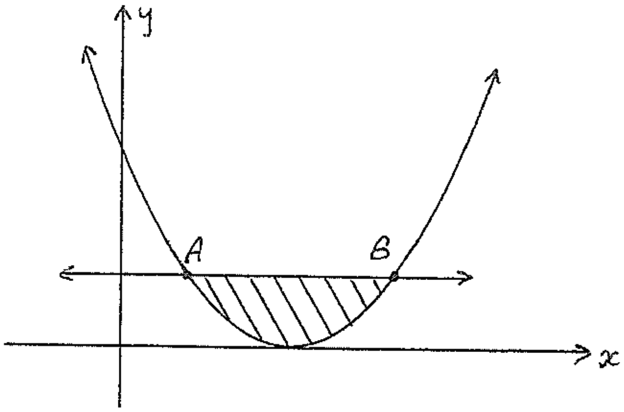
ii) Evaluate $\int_0^{\frac{\pi}{8}} 2\cos 4x \, dx$ 2

iii) On the same axes as i), draw the line $y = 2x$ 1

iv) Use your graphs above to estimate the solution to $x - \cos 4x = 0$, in terms of π. 1

Question 9 (13 marks) Start on a new page.

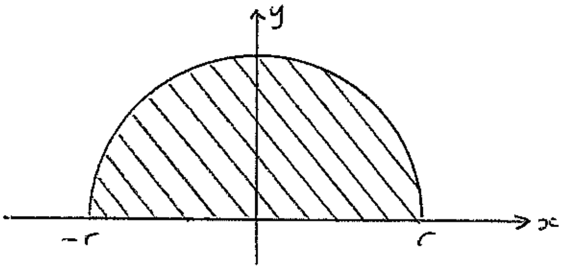
a) The area between the graphs of $y = (x-2)^2$ and $y = 1$ is shown.



i) Find x values for A and B. 2

ii) Find the shaded area. 3

b) The area between the semi-circle $y = \sqrt{r^2 - x^2}$ and the x-axis is shown.



i) Evaluate $\int_{-r}^{r} \sqrt{r^2 - x^2}\ dx$ 1

ii) The shaded area is rotated about the x-axis. Use calculus to find the exact volume thus generated. 3

c) i) Show that $\frac{d}{dx}(sin^3x) = 3\cos x - 3cos^3 x$ 2

ii) Hence find $\int cos^3 x\ dx$ 2

END OF TEST

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solutions.

① A ② D ③ E ④ C ⑤ D

⑥ a) $18°$

b) $\frac{1}{\sin \frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

c) $\tan x(\tan x - 1) = 0$

$\tan x = 0$ or 1

$\therefore x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}$

d) $y' = 3\cos 2x \times 2 = 6\cos 2x$

At $x = \frac{\pi}{12}$, $m_T = 6\cos\frac{\pi}{6} = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

e) $\left[\frac{\tan 2x}{2}\right]_{\pi/8}^{\pi/6} = \frac{\tan\frac{\pi}{3}}{2} - \frac{\tan\frac{\pi}{4}}{2} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

f)

$\text{Area} = 3 \times \int_0^{\pi} \sin x \, dx = -3\left[\cos x\right]_0^{\pi} = -3(-1-1) = 6u^2$

g) 0

⑦ a) i) $2\tan x \sec^2 x$ ii) $\frac{1}{2}\tan^2 x + c$

b) $3(\cos 5x)^2 \times -\sin 5x \times 5 = -15\cos^2 5x \sin 5x$

c) i) $\text{Area} \doteq 2 \times \left[\frac{1-0}{2}(4+3) + \frac{2-1}{2}(3+0)\right] = 2 \times \frac{1}{2}(10) = 10u^2$

ii) $\text{Area} = 2 \times \int_0^2 (4-x^2)\,dx = 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2(8 - 2\tfrac{2}{3}) - 0 = 2 \times 5\tfrac{1}{3} = 10\tfrac{2}{3}u^2$

iii) $x^2 = 4 - y$

$\therefore \text{Vol} = \pi\int_0^4 (4-y)\,dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16-8) - 0 = 8\pi u^3$

d) $\frac{d}{dx}(\tan x) = 2$

$\therefore \sec^2 x = 2$

$\frac{1}{\cos^2 x} = 2$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm\frac{1}{\sqrt{2}}$

$\therefore x = \frac{\pi}{4}$ (say)

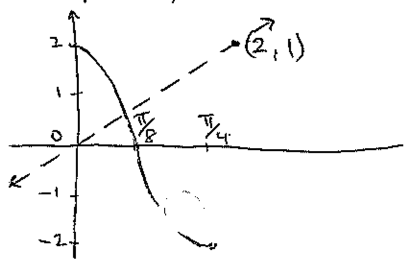
⑧ a) i) $A = \left|\int_{-1}^{1}(x^2-1)\,dx\right| + \int_1^2 (x^2-1)\,dx$

or $2\left|\int_0^1 (x^2-1)\,dx\right| + \int_1^2 (x^2-1)\,dx$

ii) $A = \left|\int_0^3 (x^2 - 2x - x)\,dx\right| = \left|\int_0^3 (x^2 - 3x)\,dx\right|$ or $\int_0^3 (3x - x^2)\,dx$

b) $\text{Area} \doteq \frac{2}{3}(0 + 4\times3 + 2\times5 + 4\times5 + 3) = \frac{2}{3}(45) = 30m^2$

c) i) amp. $= 2$, period $= \frac{2\pi}{4} = \frac{\pi}{2}$



ii) $\int_0^{\pi/8} 2\cos 4x\,dx = \left[\frac{\sin 4x}{2}\right]_0^{\pi/8} = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$

iii) on graph

iv) same as solving $2x = 2\cos 4x$ ie. approx. $x = \frac{\pi}{9}$ (say)

⑨ a) i) $(x-2)^2 = 1$

$x^2 - 4x + 4 = 1$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$\therefore x = 1$ or 3

ii) Area = rectangle − area under curve

$= 2 \times 1 - \int_1^3 (x-2)^2\,dx$

$= 2 - \left[\frac{(x-2)^3}{3}\right]_1^3$

$= 2 - \left(\frac{1}{3} - -\frac{1}{3}\right)$

$= 2 - \frac{2}{3}$

$= 1\tfrac{1}{3}u^2$

b) i) $A = \frac{1}{2}$ circle

$= \frac{\pi r^2}{2}$

ii) $Vol = 2\pi \int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx$

$= 2\pi \int_0^r (r^2 - x^2)\, dx$

$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$

$= 2\pi \left[\left(r^3 - \frac{r^3}{3}\right) - (0 - 0) \right]$

$= 2\pi \times \frac{2r^3}{3}$

$= \frac{4\pi r^3}{3} \quad u^3$

c) i) $\frac{d}{dx}\left[(\sin x)^3\right] = 3(\sin x)^2 \times \cos x$

$= 3 \sin^2 x \cos x$

$= 3(1 - \cos^2 x) \cos x$

$= 3 \cos x - 3 \cos^3 x$ as reqd.

ii) $3 \cos^3 x = 3 \cos x - \frac{d}{dx}(\sin^3 x)$

$\therefore \cos^3 x = \cos x - \frac{1}{3} \frac{d}{dx}(\sin^3 x)$

$\therefore \int \cos^3 x \, dx = \int \cos x \, dx - \frac{1}{3} \int \frac{d}{dx}(\sin^3 x)\, dx$

$= \sin x - \frac{1}{3} \sin^3 x + c$